Phase-Locking of Semiconductor Injection Lasers

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Phase locking of several semiconductor injection lasers via mutual coupling is a possible method for coherent power combination. In this report the equations describing an array of semiconductor injection lasers are formulated, their solution is outlined and the conditions needed for locking (synchronization) are derived in terms of both phenomenological and actual device parameters. It is found that for real devices these conditions are at present quite stringent but not impossible to fulfill.

I. Introduction

Semiconductor injection lasers are an attractive candidate for emitters in optical communication systems for deep space missions (Ref. 1). However, their advantages of small size, high reliability, good power efficiency and the possibility to directly modulate them at high rates are offset by the fact that the power levels emitted by a single device into a stable radiation pattern are too low for this application. One possible method for overcoming this problem is by the coherent combination of several semiconductor lasers, in a manner similar to a phased-array of antennas (Ref. 2).

There are several advantages of phase locking lasers. First, when the power of the lasers is combined incoherently, each laser emits light in its own individual spectrum. Thus it is necessary to have an optical filter with a wider bandwidth at the receiver, with a resulting increase in detected background noise radiation. Secondly, the locking of the laser components causes a reduction in the far-field beam divergence angle. This makes the task of subsequent beam narrowing simpler (e.g., by requiring an optical telescope with a smaller magnification).

The purpose of this report is to review the subject of mutual phase-locking (or synchronization) of semiconductor injection lasers. In Section II the equations of motion of a single-mode individual laser are developed, using the density matrix formulation. Section III formulates the general equations for describing an array of lasers. Then some simplifying assumptions are made and the resulting working equations are developed. Section IV outlines the solution and discusses the conditions necessary for locking in terms of phenomenological coupling coefficients. Finally, Section V analyzes the coupling coefficients in different array configurations in terms of the device parameters and geometry. These results can serve as basic guidelines for implementing various array designs.

II. Equations of Motion of a Semiconductor Laser

The equations derived in this section describe the temporal evolution of the variables pertaining to the operation of an individual, single-longitudinal-mode semiconductor injection laser. All the spatial dependence is assumed to be either uniform or averaged out. The spatial mode profile will be considered later during the calculations of the various coupling coefficients, as described in Section V. The main variables describing the laser are its electric field amplitude E, the polarization of its active medium P, and its inversion density N.

The first equation to describe the laser is the wave equation of the lasing mode, which, in cgs units can be expressed as (Ref. 5)

$$\frac{d^2E}{dt^2} + \frac{\omega}{Q_0}\frac{dE}{dt} + \omega^2 E = -4\pi\frac{d^2P}{dt^2} \tag{1}$$

where ω is the natural frequency of the laser resonator and Q_0 is the figure of merit of the laser, given by:

$$Q = \omega_0 T_{ph} \tag{2}$$

Here ω_0 is the frequency of the atomic transition, and T_{ph} is the average photon lifetime in the laser cavity.

To obtain the second equation of motion, we use the density matrix formalism of a two-level system. The semiconductor injection laser can be well approximated by a two-level system because the thermalization time within its bands is much shorter than the average interband transition times.

We start with the following equation (Ref. 3):

$$\frac{d\rho_{21}}{dt} = -i\omega_0\rho_{21} + i\frac{\mu}{\hbar}(\rho_{11} - \rho_{22})E - \frac{\rho_{21}}{T_2} \tag{3}$$

where ρ_{ij} is the ijth element in the density matrix, μ is the dipole moment of the transition, T_2 is the inelastic relaxation time and h is Planck's constant divided by 2π. A basic introduction to the subject of density matrix formalism can be found in Ref. 3, ch. 3.

Next we write the complex conjugate equation for Eq. (3), noting that $\rho_{11} - \rho_{22}, E$, and, without loss of generality, also μ, are real variables. Adding and subtracting the equations for ρ_{21} and ρ_{21}^*, a new set of first-order differential equations for $(\rho_{21} + \rho_{21}^*)$ and $(\rho_{21} - \rho_{21}^*)$ is obtained. We can further eliminate the equation for $(\rho_{21} - \rho_{21}^*)$, thus obtaining a second-order differential equation for $(\rho_{21} + \rho_{21}^*)$. Since $\rho_{21} = \rho_{12}^*$, and, by definition

$$P = \mathcal{N}\mu(\rho_{21} + \rho_{12}) \tag{4}$$

where $\mathcal{N}$ is the total population density, the resulting equation for $(\rho_{21} + \rho_{21}^*)$ can be rewritten as an equation for P. This equation is of the form

$$\frac{d^2P}{dt^2} + \frac{2}{T_2}\frac{dP}{dt} + \left(\omega_0^2 + \frac{1}{T_2^2}\right)P = -\frac{2\omega_0\mu^2}{\hbar}\mathcal{N}(\rho_{22} - \rho_{11})E \tag{5}$$

Since we are interested in the case where $\omega_0 \gg 1/T_2$ (approximately 10^{15} vs 10^9), and the inversion density is defined by

$$N = \mathcal{N}(\rho_{22} - \rho_{11}) \tag{6}$$

then the second equation can be written in its final form

$$\frac{d^2P}{dt^2} + \frac{2}{T_2}\frac{dP}{dt} + \omega_0^2 P = -\frac{2\omega_0\mu}{h}NE \tag{7}$$

The third and last equation of motion is derived starting with the following equation (Ref. 3).

$$\frac{d(\rho_{11} - \rho_{22})}{dt} = \frac{2i\mu E}{\hbar}(\rho_{21} - \rho_{21}^*) - \frac{(\rho_{11} - \rho_{22}) - (\rho_{11} - \rho_{22})_0}{T_1} \tag{8}$$

where T_1 is the elastic time constant of the transition and the subscript 0 denotes the steady-state solution. Using the equation for $(\rho_{21} + \rho_{21}^*)$ which was obtained during the derivation of Eq. (7), namely

$$\frac{d(\rho_{21} + \rho_{21}^*)}{dt} + \frac{\rho_{21} + \rho_{21}^*}{T_2} = -i\omega_0(\rho_{21} - \rho_{21}^*) \tag{9}$$

in Eq. (8), multiplying the resulting equation by $\mathcal{N}$, and using the definitions in Eqs. (4) and (6), we obtain

$$\frac{dN}{dt} + \frac{N - N_0}{T_1} = \frac{2E}{\hbar\omega_0}\left(\frac{dP}{dt} + \frac{P}{T_2}\right) \tag{10}$$

where the steady-state influx of incoming carriers, N_0, is defined as

$$N_0 = \mathcal{N}(\rho_{22} - \rho_{11})_0 \tag{11}$$

Since the polarization P oscillates at a frequency near ω_0, we can assume that $dP/dt \approx \omega_0 P >> P/T_2$, and thus we can finally write

$$\frac{dN}{dt} + \frac{N - N_0}{T_1} = \frac{2E}{\hbar\omega_0}\frac{dP}{dt} \tag{12}$$

Equations (1), (7) and (12) are the equations of motion describing the time evolution of the laser field, polarization and inversion density, respectively.

Before we continue, it is useful to convert all the variables into dimensionless parameters. The transformations are listed in Table 1, and the resulting set of dimensionless equations, equivalent to Eqs. (1), (7) and (12), is

$$\frac{d^2x}{dt'^2} + b\frac{dx}{dt'} + \Omega x = -\gamma\frac{d^2v}{dt'^2} \tag{13}$$

$$\frac{d^2v}{dt'^2} + \delta_2\frac{dv}{dt'} + v = \frac{\delta_2}{2}xw \tag{13b}$$

$$\frac{dw}{dt'} + \frac{\delta_1}{2}w = -\delta_1 - 2\delta_2 x\frac{dv}{dt'} \tag{13c}$$

Note that the time derivatives are now taken with respect to the dimensionless time $t' = \omega_0 t$, and that δ_1, δ_2, b, γ and Ω are much smaller than unity.

III. Equations Describing an Array of Lasers

In this section we consider the case of M semiconductor injection lasers which we coupled among themselves. Since the magnitude of the coupling is small under virtually all practical situations, we need to consider only interactions between each laser in the array and its closest neighbors. Furthermore, if we assume that we have an array which is arranged in a one-dimensional configuration, then each laser has two closest neighbors, except those at the extreme locations which have only one. Extension of the following analysis to the more general case is straightforward, although the calculations become more cumbersome.

In light of the above, Eqs. (13) are modified as follows in order to include the coupling terms (Ref. 4):

$$\ddot{x}_i(t') + b_i\dot{x}_i(t') + \xi_{i,i-1}\dot{x}_{i-1}(t' - \tau'_{i,i-1}) + \xi_{i,i+1}\dot{x}_{i+1}(t' - \tau'_{i,i+1}) + \Omega_i x_i(t') = -\gamma_i\ddot{v}_i(t') \tag{14a}$$

$$\ddot{v}_i(t') + \delta_2\dot{v}_i(t') + v_i(t') = \frac{\delta_2}{2}w_i(t')[x_i(t') + \eta_{i,i-1}x_{i-1}(t' - \tau'_{i,i-1}) + \eta_{i,i+1}x_{i+1}(t' - \tau'_{i,i+1})] \tag{14b}$$

$$\dot{w}_i(t') + \frac{\delta_1}{2}w_i(t') = -\delta_1 - 2\delta_2\dot{v}_i(t')[x_i(t') + \eta_{i,i-1}x_{i-1}(t' - \tau'_{i,i-1}) + \eta_{i,i+1}x_{i+1}(t' - \tau'_{i,i+1})] \tag{14c}$$

where the subscripts i, $i-1$, $i+1$ index the lasers in the array and the dots denote differentiation with respect to the dimensionless time t'; ξ_{ij} and η_{ij} are parameters associated with the coupling between the ith and the jth laser via their electric fields, and τ'_{ij} is the effective dimensionless delay time in the field coupling between the ith and the jth lasers.

It is important to note that lasers that are located at the edge of the array (e.g., $i = 1$) interact only with one other laser (since, for example x_0 does not exist). We can also see how Eqs. (14) can be extended to the more general configurations, simply by adding the effects of other lasers as needed to each equation, with the use of the proper coupling coefficients. Several conditions can be assumed in order to simplify the problem.

First, if the delay time is short enough, it can be neglected. This condition is formulated as follows (Ref. 4):

$$\omega_o\tau_{ij} = \tau'_{ij} << \frac{1}{\sqrt{\xi_{ij}^2 - \dfrac{(\omega_i - \omega_j)^2}{\omega_0^2}}} \tag{15}$$

or, equivalently

$$\tau_{ij} << \frac{1}{\sqrt{(\omega_0\xi_{ij})^2 - (\omega_i - \omega_j)^2}} \tag{16}$$

The worst case in satisfying the condition in Eq. (16) is when $\omega_i = \omega_j$. Assuming this and also that

$$\tau_{ij} \cong \frac{d_{ij}}{c/n} \tag{17}$$

where d_{ij} is the distance between the ith and jth lasers and n is the index of refraction of the medium, and noting that $\omega_0 = 2\pi c/\lambda$ (λ is the vacuum wavelength), the condition for negligible delay time is

$$\xi_{ij} \ll \frac{\lambda}{2\pi d_{ij} n} \tag{18}$$

which is satisfied under almost all practical situations.

The second simplification involves neglecting the coupling between the laser fields via the active medium. This condition is formulated as

$$\delta_2 \eta_{ij} \ll \xi_{ij} \tag{19}$$

In the following it will be assumed that the conditions expressed in Eqs. (18) and (19) are fulfilled. The consequences of violating them will be mentioned in the next section. It will also be assumed that the elements of the array are identical and equispaced, so that $\xi_{i,i\pm1} = \xi$, $b_i = b$, and $\gamma_i = \gamma$.

Under all the above assumptions, the working formulas for describing an array of coupled lasers are:

$$\ddot{x}_i(t') + b\dot{x}_i(t') + \xi\,[\dot{x}_{i-1}(t') + \dot{x}_{i+1}(t')] + \Omega_i x_i(t') = -\gamma\ddot{v}_i(t') \tag{20a}$$

$$\ddot{v}_i(t') + \delta_2\dot{v}_i(t') + v_i(t') = \frac{\delta_2}{2}\, w_i(t')\, x_i(t') \tag{20b}$$

$$\dot{w}_i(t') + \frac{\delta_1}{2} w_i(t') = -\frac{\delta_1}{2} - 2\delta_2\, \dot{v}_i(t')\, x_i(t') \tag{20c}$$

IV. Solution of the Equations

For solving the set of Eqs. (20), the "fast" temporal behavior is first factored out; i.e., we assume that

$$x_i = X_i \cos(t' + \phi_i) \tag{21a}$$

$$v_i = V_i \cos(t' + \psi_i) \tag{21b}$$

$$w_i = W_i = \text{const.} \tag{21c}$$

where the last condition implies that the spectrum of the laser driving signal lies well below its resonant frequency, which is typically a few gigahertz. Substituting Eqs. (21) into Eqs. (20), results in, after a considerable amount of algebraic manipulation and keeping only first-order terms under consideration, the following set of first-order nonlinear differential equations for the "slow" amplitudes of the variables:

$$\dot{X}_i = -\frac{b}{2} X_i - \frac{\xi}{2}\,[X_{i+1}\cos(\phi_{i+1} - \phi_i) + X_{i-1}\cos(\phi_{i-1} - \phi_i)] + \frac{\gamma V_i}{2}\sin(\psi_i - \phi_i) \tag{22a}$$

$$\dot{\phi}_i = \frac{1}{2}(\Omega_i - 1) - \frac{\xi}{2X_i}\,[X_{i+1}\sin(\phi_{i+1} - \phi_i) + X_{i-1}\sin(\phi_{i-1} - \phi_i)] - \frac{\gamma}{2}\frac{V_i}{X_i}\cos(\psi_i - \phi_i) \tag{22b}$$

$$\dot{V}_i = -\frac{\delta_2 V_i}{2} - \frac{\delta_2}{4} X_i W_i \sin(\psi_i - \phi_i) \tag{22c}$$

$$\dot{\psi}_i = -\frac{\delta_2}{4}\frac{X_i W_i}{V_i}\cos(\psi_i - \phi_i) \tag{22d}$$

$$\dot{W}_i = -\frac{\delta_1}{2} W_i - \delta_1 + \delta_2 X_i V_i \sin(\psi_i - \phi_i) \tag{22e}$$

The first subject to be investigated is the steady-state solution of the above equations. For this we let $\dot{X}_i = \dot{V}_i = \dot{W}_i = 0$ (i.e., constant amplitudes) and $\dot{\phi}_i = \dot{\psi}_i \equiv \delta\omega$ (i.e., phase locking) in Eqs. (22). The resulting set of nonlinear algebraic equations describing the steady-state array variables is given by:

$$X_i\left[b - \frac{\gamma \sin^2(\psi_i - \phi_i)}{1 + \frac{\delta_2}{\delta_1} X_i^2 \sin^2(\psi_i - \phi_i)}\right] = -\xi\,[X_{i+1}\cos(\phi_{i+1} - \phi_i) + X_{i-1}\cos(\phi_{i-1} - \phi_i)] \tag{23a}$$

$$2\delta\omega + (1 - \Omega_i) + \gamma\frac{\sin(\psi_i - \phi_i)\cos(\psi_i - \phi_i)}{1 + \frac{\delta_2}{\delta_1} X_i^2 \sin^2(\psi_i - \phi_i)} = -\frac{\xi}{X_i}[X_{i+1}\sin(\phi_{i+1} - \phi_i) + X_{i-1}\sin(\phi_{i-1} - \phi_i)] \tag{23b}$$

$$\delta\omega = \frac{\delta_2}{2}\cos(\psi_i - \phi_i) \tag{23c}$$

Since one of the ϕ_i's (or ψ_i's) can be set arbitrarily to zero, and in all the lasers $(\psi_i - \phi_i)$ is the same, Eqs. (23) is a set of $2M$ equations with $2M$ unknowns, where M is the number of lasers in the array. Once Eqs. (23) are solved (which can be done only numerically), the other variables can be expressed in terms of the X_i's, ϕ_i's and ψ_i's, as follows:

$$W_i = -\frac{2}{1 + \frac{\delta_2}{\delta_1} X_i^2 \sin^2(\psi_i - \phi_i)} \tag{24}$$

and

$$V_i = \frac{X_i \sin(\psi_i - \phi_i)}{1 + \frac{\delta_2}{\delta_1}[X_i \sin(\psi_i - \phi_i)]^2} \tag{25}$$

The uncoupled (i.e., $\xi = 0$) solution of Eqs. (23) – (25) is obtained straightforwardly. In this case the phase shifts between the ϕ_i's are undefined, the phase shift in each laser between its field and its polarization is

$$\tan(\psi_i^0 - \phi_i^0) = \left[\frac{b + \delta_2}{\Omega_i - 1}\right] \tag{26}$$

approaching $\pi/2$ as the natural frequency of the resonator is closer to the transition frequency, and the field amplitude is given by

$$|X_i^0| = \sqrt{\frac{\delta_1}{\delta_2}}\sqrt{\frac{\gamma}{b} - \left(\frac{\Omega_i - 1}{b + \delta_2}\right)^2 - 1} \tag{27}$$

where the pumping term is contained in γ (see Table 1). In Appendix A the relation between the formalism used in this report and the solutions obtained from the laser rate equations is discussed.

Since the phenomenon of locking is a nonlinear one, we cannot solve for the coupled case by using perturbation methods on the uncoupled case, especially since the phase relationships between the lasers have a sudden transition as locking occurs. However, several other numerical methods, such as successive iterations, can be used to yield the desired results.

If we sum up Eq. (23b) for $i = 1, 2, \ldots M$, we find that if ω_0 is the "center of mass" of all the uncoupled lasers' frequencies, i.e.,

$$\sum_{i=1}^{m} X_i^2 (1 - \Omega_i) = 0 \tag{28}$$

then we have

$$\psi_i - \phi_i = \frac{\pi}{2} \tag{29}$$

and the synchronized array oscillates at the center transition frequency ω_0.

As shown in Refs. 4–6, all but one of the conditions for stability are identical to the conditions of existence of a solution. The stability investigation is done on the equations of the "slow" variables in Eqs. (22), with the derivative of the "fast" variables set to zero (since they are "fast" they can always follow adiabatically the "slow" variables). For semiconductor injection lasers with common parameters, the following conditions hold:

$$T_2, T_{ph} >> T_1 \tag{30}$$

and thus W_i and X_i are the "slow" variables and V_i are the "fast" variables. An exact stability analysis of this nonlinear problem is prohibitively complicated, so a linearized sensitivity test about the steady-state solution is carried out, as outlined in Refs. 4 and 5.

To conclude, we can say that if a solution exists, and if the figure of merit of the laser resonator increases when becoming part of the array, the solution is stable. In the following, these stability conditions will be stated more explicitly.

Successive application of Eq. (23b), using Eqs. (28) and (29), yields the following expression for the sine of the phase shift between two neighboring lasers:

$$\sin(\phi_{k+1} - \phi_k) = -\frac{1}{\xi X_{k+1} X_k} \sum_{i=1}^{k} X_i^2 (1 - \Omega_i)$$

$$i = 1, 2, \ldots (M - 1) \tag{31}$$

Since the sine value of a real angle is always bounded by unity, the right-hand side of Eq. (31) must be bounded by unity as expressed in the following equation:

$$\left|\sum_{i=1}^{k} X_i^2 (1 - \Omega_i)\right| < \left|\xi X_{k+1} X_k\right|$$

$$i = 1, 2, \ldots (M - 1) \tag{32}$$

When the coupling is not too strong, then to a first approximation $X_1 \cong X_2 \cong \ldots X_M$, and the condition in Eq. (32) simplifies to

$$\left|\sum_{i=1}^{k} (1 - \Omega_i)\right| < \left|\xi\right| \; i = 1, 2, \ldots (M - 1) \tag{33}$$

The second stability condition requires that the intensity of the field (i.e., X^2) is to be a positive number. From Eq. (23a) (using again Eqs. (28) and (29)) this condition can be expressed as

$$X_k \frac{\gamma - b}{\xi} > X_{k+1} \cos(\phi_{k+1} - \phi_k) + X_{k-1} \cos(\phi_k - \phi_{k-1})$$

$$k = 1, 2, \ldots M \tag{34}$$

or, using Eq. (31):

$$X_k^2 (\gamma - b) > \sqrt{(\xi X_{k+1} X_k)^2 - \left[\sum_{i=1}^{k} X_i^2 (1 - \Omega_i)\right]^2}$$

$$+ \sqrt{(\xi X_k X_{k-1})^2 - \left[\sum_{i=1}^{k-1} X_i^2 (1 - \Omega_i)\right]^2}$$

$$k = 1, 2, \ldots M \tag{35}$$

Again, this condition simplifies considerably in the region where the coupling is not strong, and thus $X_1 \cong X_2 \cong \ldots X_M$. In that case the condition in Eq. (35) simplifies to

$$\gamma - b > \sqrt{\xi^2 - \left[\sum_{i=1}^{k} (1 - \Omega_i)\right]^2} + \sqrt{\xi^2 - \left[\sum_{i=1}^{k-1} (1 - \Omega_i)\right]^2}$$

$$k = 1, 2, \ldots m. \tag{36}$$

The last stability condition comes from energetic consideration. The laser needs to "gain" something by "joining" the array; i.e., its Q needs to increase. This implies that

$$\xi < 0 \tag{37}$$

Equations (33), (36) and (37) summarize the approximate conditions for existence and stability of a solution for an array of semiconductor lasers. It should be noted, however, that as the coupling constant ξ becomes larger and approaches the value of $b(= 1/Q)$, these approximations cease to be accurate and Eqs. (32) and (35) have to be used instead of Eq. (33) and (36).

The reason is that when ξ is comparable to b, the change in $1/Q$ due to the external interaction is comparable to the intrinsic $1/Q$ of the resonator. In this limit of very strong coupling, each individual laser loses its identity almost completely, and the intensities of each laser vary considerably, being strongly influenced by the phase-shifts among the lasers. In this region it is more relevant to analyze the whole array as one "super waveguide" (Ref. 7). In most practical cases, however, the limit of very strong coupling is not reached.

V. Coupling Coefficients in Different Configurations

In the preceding section the conditions for obtaining phase locking of an array of lasers were derived in terms of phenomenological coupling parameters (ξ). In this section we will relate these coupling parameters to the actual device parameters. Most of the section will discuss mutual coupling between lasers via field interaction due to their close proximity. The subject of diffraction coupling will also be briefly discussed.

Coupling between two semiconductor lasers which are in close proximity to each other is equivalent to coupling between two waveguides (see Fig. 1). The coupling strength commonly used in the literature is defined as coupling per unit length and is denoted by K (cm^{-1}). We first want to establish the relation between K and the coupling parameter ξ used in the wave equation. By inspection of Eq. (14a) we see that ξ_{ij} is the change in the effective $1/Q$ of the ith laser in the array due to its interaction with the jth laser. The fraction of the field lost during one optical cycle is π/Q, so the fractional field coupled during the time of one optical cycle is $\pi\xi$. The fraction of the field coupled when the wave propagates a distance of 1 cm is $|K|$. The wave travels this distance in a time that is equivalent to n/λ optical cycles, where λ is the vacuum wavelength of the radiation and n is the index of refraction of the material.

Thus, we can conclude that

$$\pi\xi = \frac{\lambda}{n} K \tag{38}$$

or

$$\xi = \frac{2K}{k} \tag{39}$$

where $k = 2\pi n/\lambda$.

In order to calculate the coupling coefficient K between two semiconductor lasers in close proximity, we must solve first for the eigenmodes and the eigenvalues of a waveguide which is made of a region with an index of refraction n_2 which is surrounded by two semi-infinite regions with an index of refraction n_1 $(n_1 < n_2)$. Such a waveguide is depicted in Fig. 2. By using the effective index formalism (Ref. 9) we can reduce the two-dimentional problem of a laser waveguide cross-section (see Fig. 1) to the one-dimensional waveguide depicted in Fig. 2.

The transverse profile of the eigenmodes field is given by

$$E(x) = \begin{cases} \cos(hx) & |x| < \frac{s}{2} \quad (40a) \\ \cos\left(\frac{hs}{2}\right) e^{q\frac{s}{2}} e^{-q|x|} & |x| > \frac{s}{2} \quad (40b) \end{cases}$$

where

$$h^2 \equiv n_2^2 k_0^2 - \beta^2 \tag{41a}$$

$$q^2 \equiv \beta^2 - n_1^2 k_0^2 \tag{41b}$$

and β is the propagation constant eigenvalue of the mode (see below). The general solution for the eigenvalue involves a transcendental equation (Ref. 3, ch. 19). However, an approximate analytical solution is possible in the following two cases.

(1) Weakly guided modes; i.e.,

$$n_2^2 - n_1^2 \ll \left(\frac{\lambda}{2s}\right)^2 \tag{42}$$

In this case

$$h \cong k\sqrt{n_2^2 - n_1^2} \tag{43a}$$

$$q \cong \frac{\left(n_2^2 - n_1^2\right) k^2 s}{2} \tag{43b}$$

$$\beta \approx n_1 k \tag{43c}$$

$$q \ll h \ll \frac{\pi}{s} \tag{43d}$$

(2) Well-confined modes; i.e.,

$$n_2^2 - n_1^2 \gg \left(\frac{\lambda}{2s}\right)^2 \tag{44}$$

In this case

$$h \cong \frac{\pi}{s} \tag{45a}$$

$$q \cong k\sqrt{n_2^2 - n_1^2} \tag{45b}$$

$$\beta \cong n_2 k \tag{45c}$$

$$q \gg h \cong \frac{\pi}{s} \tag{45d}$$

Next we assume that the field propagating in the lasers in one direction is of the form

$$\mathcal{E}(x, z, t) = E(x, z)\, e^{i(\omega t - \beta Z)} \tag{46}$$

This leads, after using the usual adiabatic approximation, to the following wave equation

$$\frac{d^2E}{dx^2} = 2i\beta \frac{dE}{dZ} + (\epsilon k^2 - \beta^2) E = 0 \tag{47}$$

We assume that the field growth along the Z direction is due to net gain g in its own active medium as well as the coupling K from the field of the other laser, and that the two interacting waves propagate codirectionally. The maximum coupling occurs when the propagation constants in the two lasers are identical (i.e., phase-matching), in which case we can write (Ref. 3, ch. 16):

$$\frac{dE_1}{dZ} = gE_1 - iK E_2 \tag{48}$$

We define the complex index of refraction as

$$\tilde{\epsilon} = \epsilon + 2i \frac{\beta}{k^2} g \tag{49}$$

Using Eqs. (48) and (49) or (47) results in the following equation for the transverse profile of the field:

$$\frac{d^2E_1}{dx^2} + (\tilde{\epsilon}k^2 - \beta^2)E_1 + 2\beta K E_2 = 0 \qquad (50)$$

Multiplying Eq. (50) by E_2^* and integrating (also integration by parts) yields the following expression for K

$$K = \frac{\int\left(\frac{dE_1}{dx}\right)\left(\frac{dE_2}{dx}\right)^* dx - \int\left(\tilde{\epsilon}k^2 - \beta^2\right)E_1 E_2^* dx}{2\beta\int|E_2|^2 dx} \qquad (51)$$

Since the coupling is small, we can take for E_1 and E_2 the unperturbed field profiles, as given by Eq. (40) for E_1 and its displaced version for E_2.

For the case of weakly guided modes (Eq. (42)) we obtain

$$K = -\frac{\left[s\left(n_2^2 - n_1^2\right)\right]^2 k^3}{8\sqrt{\epsilon}} e^{-\frac{s(n_2^2 - n_1^2)k^2 d}{2}} \qquad (52)$$

where d is the separation between the two lasers (see Fig. 1) and $\sqrt{\epsilon} = n_2 \cong n_1$. Note that K, and hence ξ, are negative, as required (Eq. (37)).

This coupling has a maximum when

$$s\left(n_2^2 - n_1^2\right) = \frac{4}{k^2 d} \qquad (53)$$

and the resulting optimum coupling is

$$|K|_{max} = \frac{\lambda_0}{\pi e^2 d^2\sqrt{\epsilon}} \qquad (54)$$

Using the relationship between K and ξ (Eq. 30), we can write:

$$|\xi|_{max} = \left(\frac{2}{\sqrt{\epsilon}\pi e d}\right)^2 \qquad (55a)$$

If we take GaAs as a typical example, then we have $\lambda = 0.9\ \mu m$ and $\sqrt{\epsilon} \cong 3.6$, in which case we obtain

$$|\xi|_{\substack{max \\ (GaAs)}} \cong 8 \cdot 10^{-4}\frac{1}{d^2} \qquad (55b)$$

where d is expressed in micrometers.

For well-confined modes (Eq. 36), the coupling is given by

$$|K| = \frac{\lambda^2}{2\sqrt{\epsilon}\sqrt{n_2^2 - n_1^2}\,s^3} e^{-kd\sqrt{n_2^2 - n_1^2}} \qquad (56)$$

and is usually much smaller than the coupling in the other case (Eqs. 52, 54).

As an example of the application of Eq. (55) to the design of laser arrays, let us assume that the differences in the natural oscillation frequencies of the individual lasers are due to different lengths

$$\frac{(1 - \Omega_i)}{2} = \frac{\Delta\omega_i}{\omega_0} = -\frac{\Delta L_i}{L_i} \qquad (57)$$

where L_i is the length of the cavity of the ith laser. From Eqs. (33), (55) and (57) we find that

$$\left|\sum_{i=1}^{k}\Delta L_i\right| \lesssim 5 \cdot 10^{-4} L\left(\frac{\lambda}{d}\right)^2 \quad i = 1, 2, \ldots M \qquad (58)$$

when $L_i \cong L_2 \cong L_i = L$. For $L \cong 300\ \mu m$ and $\lambda/d \cong 0.1$, we obtain

$$\left|\sum_{i=1}^{k}\Delta L_i\right| \lesssim 15\ \text{Å}$$

which is a rather stringent requirement. For $(\lambda/d) \sim 0.2$, the above constraint is increased to 60 Å.

We can now check the validity of the approximation of Eq. (19) by neglecting the coupling of the fields via the active medium. Assuming weakly guided modes (which provide for stronger coupling) and that a uniform carrier profile exists at the laser stripe of a width s as well as an additional region of width $s_0/2$ on both of its sides, we obtain

$$\eta = q(s + s_0)e^{-qd} \qquad (59)$$

Using Eqs. (33), (43b) and (52), we see that the condition expressed in Eq. (19) is fulfilled if the following condition holds

$$\frac{\Delta n}{n} >> \delta_2 \tag{60}$$

where $\Delta n = n_2 - n_1$. The condition implied in Eq. (60) states that the fractional change of the index of refraction across the laser structure is much larger than the normalized spontaneous carrier lifetime, and this condition holds in most cases. If necessary, the effect of η can be included by replacing $|\xi|$ by $|\xi| + \delta_2|\eta|$ in all the stability conditions. We see that the increased coupling via the active region allows phase-matching for greater frequency deviations.

Finally, we will briefly summarize the case where we have diffraction coupling. The lasers are put in an external cavity, and part of the field of each laser is reflected into the other lasers producing a coupling between them. The magnitude of the coupling coefficient in this case has been determined in Refs. 5, 6, and 8. The general calculations in this case do not yield simple analytical expressions (Refs. 5, 8). However, if we assume that the sides of the lasers facing the external mirror are coated with an anti-reflection coating and that neighboring lasers can interact with one another, i.e.,

$$\theta_{F\cdot F} > \frac{d+s}{L_{ext}} \tag{61}$$

where $\theta_{F\cdot F}$ is the far-field angle of the laser radiation pattern in the plane of the array configuration, and L_{ext} is the distance from the end face of the laser to the external mirror (Fig. 3), then the coupling constant is given by (Ref. 6).

$$\xi_{diff} = -\frac{1}{2}\left(\frac{\lambda}{2\pi}\right)^3 \frac{1}{Ls(s+d)(n_2^2 - n_1^2)} \tag{62}$$

where L is the total length of the resonator. The coupling constant is inversely proportional to the distance between the lasers, to the length of the resonator and to the stripe width of each laser. The last functional relationship is due to the fact that the far-field angle of the laser radiation is inversely proportional to its stripe width.

Dividing Eqs. (55) and (62) we find that the ratio between the two is approximately

$$\frac{\xi_{\textit{fields overlap}}}{\xi_{\textit{diffraction}}} = \left(\frac{L}{\lambda/n}\right)\frac{s}{d}\left(1+\frac{s}{d}\right)\Delta n \tag{63}$$

Depending on the array parameters, one of the two coupling mechanisms is dominant. For example, from Eq. (63) we see that as the separation d between the individual lasers increases, there is more coupling due to diffraction than due to field overlap (note, however, that both couplings are at least inversely proportional to d).

VI. Conclusions

The problem of phase-locking of several semiconductor injection lasers via mutual coupling has been investigated. First, the equations of motion of the parameters of both individual lasers and an array of lasers were derived, considering both coupling via field overlapping and interactions via the laser active media. The solution of the equations was outlined and, more important, the conditions for obtaining phase-matching were derived. These conditions put restrictions on both the sign and the magnitude of the coupling constraint: it must be negative so that phase-matched (synchronized) operation is energetically favorable, and its magnitude must be greater than the standard deviation of the lasers' frequency deviations from the laser transition frequency.

Finally, the actual magnitudes of the coupling coefficients in several configurations were calculated, and the results indicate that it is possible, although not trivial, to achieve the coupling conditions.

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Table 1. List of transformations into dimensionless parameters

	Old variable	New dimensionless parameter	Transformation
Time	t	t'	$t' = \omega_0 t$
Electric field	E	x	$x = \frac{\mu T_2}{\hbar} E$
Polarization	P	v	$v = \frac{1}{\mu N_0} P$
Inversion density	N	w	$w = -\frac{2}{N_0} N$
Pump density	N_0	γ	$\gamma = \frac{4\pi\mu^2}{\hbar} T_2 N_0$
Elastic rate of the atomic transition (transition linewidth)	T_1^{-1}	δ_1	$\delta_1 = \frac{2}{\omega_0 T_1}$
Inelastic rate of the atomic transition (corresponds to the carrier's lifetime)	T_2^{-1}	δ_2	$\delta_2 = \frac{2}{\omega_0 T_2}$
Figure of merit of laser resonator	Q	b	$b = \frac{1}{Q}\left(\frac{\omega}{\omega_0}\right)$
Frequency of laser oscillation	ω	Ω	$\Omega = \left(\frac{\omega}{\omega_0}\right)^2$

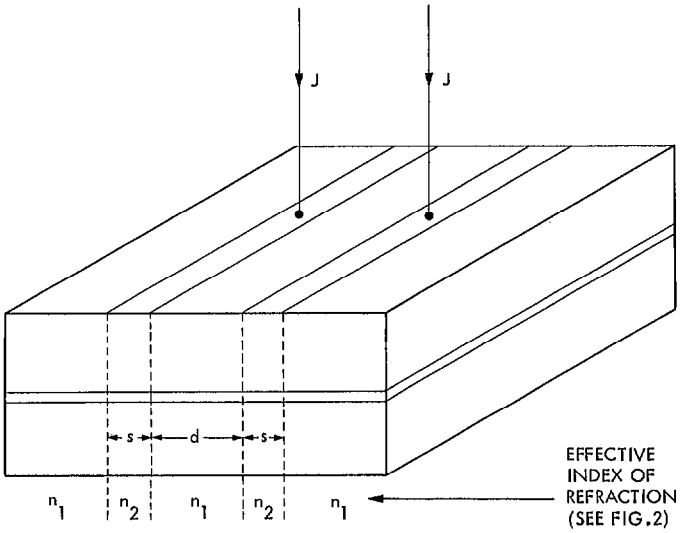

Fig. 1. Schematic configuration of two semiconductor lasers in close proximity

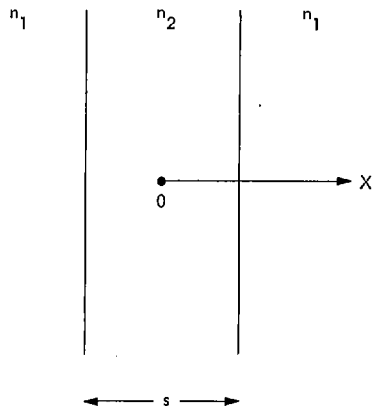

Fig. 2. Schematic drawing of a one-dimensional symmetric slab waveguide

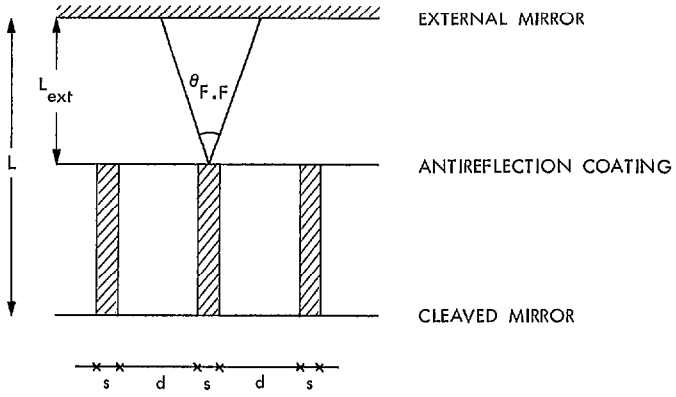

Fig. 3. Schematic configuration of several semiconductor lasers in an external resonator

Appendix A

Relation Between Parameters in this Article and Parameters Appearing in the Laser Rate Equations

The linearized steady-state rate equations can be written as (Ref. 10)

$$0 = \frac{J}{qd} - ANS - \frac{N}{\tau_s} \tag{A-1}$$

$$0 = ANS - \frac{S}{\tau_{ph}} \tag{A-2}$$

where N and S are the carrier and photon densities, respectively, τ_s and τ_{ph} are the carrier and photon lifetimes, respectively, A is the gain coefficient, J is the current density, q is the electron charge and d is the thickness of the active region.

Solution of Eqs. (A-1), (A-2) yield

$$N = \frac{\frac{J\tau_s}{qd}}{1 + A\,\tau_s\,S} \tag{A-3}$$

$$A\,\tau_s\,S = \frac{J\,A\,\tau_s\,\tau_{ph}}{qd} - 1 \tag{A-4}$$

From Eqs. (24), (29) and Table 1 we obtain

$$N = \frac{N_0}{1 + \frac{T_1}{T_2}X^2} \tag{A-5}$$

Thus, we can identify N_0 with

$$N_0 = \frac{J\,\tau_s}{qd} \tag{A-6}$$

and

$$\frac{T_1}{T_2}X^2 = A\,\tau_s\,S \tag{A-7}$$

Since by definition $X^2 = E^2/h\omega_0$ and $T_2 = \tau_s$, we also obtain the following relations:

$$A = 4\pi\,\frac{\mu^2\omega_0 T_2}{h} \tag{A-8}$$

and

$$\gamma = \frac{A N_0}{\omega_0} \tag{A-9}$$

Thus we see that the rate equations formalism and the formalism used in this report are equivalent when applied to individual lasers. Of course, the rate equations are just particle bookkeeping equations. They do not describe the electromagnetic wave and thus cannot be used in formulating the synchronization problem.